Systems of linear equations

Miroslav Kureš

Solving systems of linear equations by hand needs to be done thoughtfully. We give examples from which it will be clear what is meant by this.

Example 1. Let us solve the following system in $\mathbb{C}$:

$$2a + 3b + \mathrm{i}d = 27 + 13\mathrm{i} \tag{1.1}$$

$$2a + 4\mathrm{i}b + 5c = 41 + 31\mathrm{i} \tag{1.2}$$

$$2\mathrm{i}a + b - 2d = -12 + 9\mathrm{i} \tag{1.3}$$

$$2\mathrm{i}a - \mathrm{i}c - 4\mathrm{i}d = 3 - 29\mathrm{i} \tag{1.4}$$

First of all, we note that a mechanical use of the Gauss elimination method would lead to very laborious calculations. So let us think how to arrange the rows and columns in the starting matrix! We suggest to begin with the matrix

$$\begin{array}{cccc} 2a & c & d & b \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & \mathrm{i} & 3 & 27+13\mathrm{i} \\ -1 & 1 & 4 & 0 & 29+3\mathrm{i} \\ -1 & -5 & 0 & -4\mathrm{i} & -41-31\mathrm{i} \\ -1 & 0 & -2\mathrm{i} & \mathrm{i} & -9-12\mathrm{i} \end{array}\right) \quad \begin{array}{c} (1.1) \\ \mathrm{i}(1.4) \\ (-1)(1.2) \\ \mathrm{i}(1.3) \end{array}$$

in which we add the first row to remaining

$$\begin{array}{cccc} 2a & c & d & b \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & \mathrm{i} & 3 & 27+13\mathrm{i} \\ 0 & 1 & 4+\mathrm{i} & 3 & 56+16\mathrm{i} \\ 0 & -5 & \mathrm{i} & 3-4\mathrm{i} & -14-18\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 3+\mathrm{i} & 18+\mathrm{i} \end{array}\right)$$

and then the fourth row to remaining:

$$\begin{array}{cccc} 2a & c & d & b \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 6+\mathrm{i} & 45+14\mathrm{i} \\ 0 & 1 & 4 & 6+\mathrm{i} & 74+17\mathrm{i} \\ 0 & -5 & 0 & 6-3\mathrm{i} & 4-17\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 3+\mathrm{i} & 18+\mathrm{i} \end{array}\right)$$

Now, the fourth row can be multiplied by i and interchanged with the third row:

$$\begin{array}{cccc|c} 2a & c & d & b & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 6+\mathrm{i} & 45+14\mathrm{i} \\ 0 & 1 & 4 & 6+\mathrm{i} & 74+17\mathrm{i} \\ 0 & 0 & 1 & -1+3\mathrm{i} & -1+18\mathrm{i} \\ 0 & -5 & 0 & 6-3\mathrm{i} & 4-17\mathrm{i} \end{array}\right)$$

We realize the elimination in the fourth row by adding the appropriate multiples of the second and third rows. We obtain the final row echelon matrix

$$\begin{array}{cccc|c} 2a & c & d & b & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 6+\mathrm{i} & 45+14\mathrm{i} \\ 0 & 1 & 4 & 6+\mathrm{i} & 74+17\mathrm{i} \\ 0 & 0 & 1 & -1+3\mathrm{i} & -1+18\mathrm{i} \\ 0 & 0 & 0 & -28+29\mathrm{i} & -197+146\mathrm{i} \end{array}\right)$$

and derive

$$b = \frac{-197+146\mathrm{i}}{-28+29\mathrm{i}} = \frac{-197+146\mathrm{i}}{-28+29\mathrm{i}} \cdot \frac{-28-29\mathrm{i}}{-28-29\mathrm{i}} = \frac{9750+1625\mathrm{i}}{1625} = 6+\mathrm{i},$$

which was probably the most difficult part of the calculation by hand. However, our resulting matrix is now in such a suitable form that we can complete the solution very quickly by

$$d = 8+\mathrm{i},\ c = 7+\mathrm{i}\ \text{ and } a = 5+\mathrm{i}.$$

Example 2. Let us solve the following system in $\mathbb{R}$

$$3a + b + 12c - d = 41 \tag{2.1}$$

$$3a + 118b + 18c + 15d = 40 \tag{2.2}$$

$$9a + 2b - 18c + 9d = 92 \tag{2.3}$$

$$8a - 8b - 18c + 7d = 81 \tag{2.4}$$

and, if possible, let us search for some its integer solution.

Now, we suggest to begin with the matrix

$$\begin{array}{cccc|c} c & a & d & b & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 18 & 3 & 15 & 118 & 40 \\ -18 & 9 & 9 & 2 & 92 \\ -18 & 8 & 7 & -8 & 81 \\ -36 & -9 & 3 & -3 & -123 \end{array}\right) \quad \begin{array}{c} (2.2) \\ (2.3) \\ (2.4) \\ -3(2.3) \end{array}$$

in which we add the first row to the second row and to the third row and add its double to the fourth row. (Why is c leftmost? Because that is where we eliminate by the easiest way. Why is b rightmost? Because it has the ugliest coefficient 118.) We obtain

$$\begin{array}{cccc|c} c & a & d & b & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 18 & 3 & 15 & 118 & 40 \\ 0 & 12 & 24 & 120 & 132 \\ 0 & 11 & 22 & 110 & 121 \\ 0 & -3 & 33 & 233 & -43 \end{array}\right)$$

and one can observe that the second row and the the third row are linearly dependent. We divide the second row by 12 and place the third (dependent and already zeroed) row as the last row:

$$\begin{array}{c} \begin{matrix} c & a & d & b & \end{matrix} \\ \left(\begin{array}{cccc|c} 18 & 3 & 15 & 118 & 40 \\ 0 & 1 & 2 & 10 & 11 \\ 0 & -3 & 33 & 233 & -43 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Finally, we add the triple of the second row to the third row:

$$\begin{array}{c} \begin{matrix} c & a & d & b & \end{matrix} \\ \left(\begin{array}{cccc|c} 18 & 3 & 15 & 118 & 40 \\ 0 & 1 & 2 & 10 & 11 \\ 0 & 0 & 39 & 263 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Now, we take b as an arbitrary real parameter p, i.e. $b = p$ and find that

$$d = \frac{1}{39}\left(-10 - 263p\right) \qquad \text{and} \qquad a = \frac{1}{39}\left(449 + 136p\right),$$

while the most laborious is the calculation of

$$c = \frac{1}{18 \cdot 39}\left(363 - 1065p\right) = \frac{1}{234}\left(121 - 355p\right).$$

The search for an integer solution fails if we try $p = 0$. But for $p = 1$ we are successful:.

$$a = 15, \quad b = 1, \quad c = -1 \quad \text{and} \quad d = -7$$

is an example of an integer solution of the system.