A Very Quick Introduction to Moving in Space Using Dual Quaternions

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Abstract. An instructive example for understanding how a spatial motion can be realized using dual quaternions is given.

In visualization (in particular in computer games), in robotics and in many other applications, dual quaternions enable to realize moving of objects.

Historically, in mathematics many times, algorithms for computations were known before any theory was built. Also didactically, it is sometimes more appropriate to proceed in this way. That is why this short note was written.

In our example, we will show a computation with dual quaternions. Use it, spatial transformations can be realized, including both translations and rotations.

As usually, $\mathbb{R}$ are real numbers. Further,

$$\mathbb{D} = \left\{ \mathbf{x} = x + X\varepsilon; x, X \in \mathbb{R}, \varepsilon^2 = 0 \right\}$$

are dual numbers (over $\mathbb{R}$). Quaternion operations are calculated in the usual way, especially

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \mathrm{ij} = \mathrm{k} = -\mathrm{ji};$$

dual quaternions have coefficients from $\mathbb{D}$. Using their multiplication expresses required geometric transformations.

The procedure will be quite clear from the following example.

Example 1. We will show how rotation followed by translation is calculated.

Let us consider the point $X = [4, 2, 2]$. Let us rotate it around the axis given by the (unit) vector $\vec{u} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ by the angle $\varphi = \frac{\pi}{3}$ and then, let us translate the image by the vector $\vec{v} = (-3, -2, -1)$. (The axis of rotation passes through the origin.)

We represents the point as the dual quaternion

$$\boldsymbol{\xi} = 1 + x_1\varepsilon\mathrm{i} + x_2\varepsilon\mathrm{j} + x_3\varepsilon\mathrm{k} = 1 + 4\varepsilon\mathrm{i} + 2\varepsilon\mathrm{j} + 2\varepsilon\mathrm{k}.$$

For rotation, we express the dual quaternion

$$\boldsymbol{\alpha}_r = \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}u_1\mathrm{i} + \sin\frac{\varphi}{2}u_2\mathrm{j} + \sin\frac{\varphi}{2}u_3\mathrm{k} = \frac{\sqrt{3}}{2} + \frac{1}{6}\mathrm{i} + \frac{1}{3}\mathrm{j} + \frac{1}{3}\mathrm{k}$$

and its conjugate of the first kind

$$\bar{\boldsymbol{\alpha}}_r = \cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}u_1\mathrm{i} - \sin\frac{\varphi}{2}u_2\mathrm{j} - \sin\frac{\varphi}{2}u_3\mathrm{k} = \frac{\sqrt{3}}{2} - \frac{1}{6}\mathrm{i} - \frac{1}{3}\mathrm{j} - \frac{1}{3}\mathrm{k}.$$

For translation, we express the dual quaternion

$$\boldsymbol{\alpha}_t = 1 + \frac{v_1}{2}\varepsilon\mathrm{i} + \frac{v_2}{2}\varepsilon\mathrm{j} + \frac{v_3}{2}\varepsilon\mathrm{k} = 1 - \frac{3}{2}\varepsilon\mathrm{i} - \varepsilon\mathrm{j} - \frac{1}{2}\varepsilon\mathrm{k}$$

and its conjugate of the second kind

$$\hat{\boldsymbol{\alpha}}_t = 1 + \frac{v_1}{2}\varepsilon\mathrm{i} + \frac{v_2}{2}\varepsilon\mathrm{j} + \frac{v_3}{2}\varepsilon\mathrm{k} = 1 - \frac{3}{2}\varepsilon\mathrm{i} - \varepsilon\mathrm{j} - \frac{1}{2}\varepsilon\mathrm{k}.$$

Now, we compute

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_t \cdot \boldsymbol{\alpha}_r =$$
$$\frac{\sqrt{3}}{2} + \frac{3}{4}\varepsilon + \left(\frac{1}{6} + \left(-\frac{1}{6} - \frac{3\sqrt{3}}{4}\right)\varepsilon\right)\mathrm{i} + \left(\frac{1}{3} + \left(\frac{5}{12} - \frac{\sqrt{3}}{2}\right)\varepsilon\right)\mathrm{j} + \left(\frac{1}{3} + \left(-\frac{1}{3} - \frac{\sqrt{3}}{4}\right)\varepsilon\right)\mathrm{k}$$

and

$$\tilde{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}}_r \cdot \hat{\boldsymbol{\alpha}}_t =$$
$$\frac{\sqrt{3}}{2} - \frac{3}{4}\varepsilon + \left(-\frac{1}{6} + \left(-\frac{1}{6} - \frac{3\sqrt{3}}{4}\right)\varepsilon\right)\mathrm{i} + \left(-\frac{1}{3} + \left(\frac{5}{12} - \frac{\sqrt{3}}{2}\right)\varepsilon\right)\mathrm{j} + \left(-\frac{1}{3} + \left(-\frac{1}{3} - \frac{\sqrt{3}}{4}\right)\varepsilon\right)\mathrm{k}.$$

Finally,

$$\tilde{\boldsymbol{\xi}} = \tilde{\xi}_1 + \tilde{\Xi}_1\varepsilon + \left(\tilde{\xi}_2 + \tilde{\Xi}_2\varepsilon\right)\mathrm{i} + \left(\tilde{\xi}_3 + \tilde{\Xi}_3\varepsilon\right)\mathrm{j} + \left(\tilde{\xi}_4 + \tilde{\Xi}_4\varepsilon\right)\mathrm{k} = \boldsymbol{\alpha} \cdot \xi \cdot \tilde{\boldsymbol{\alpha}} =$$
$$1 + \left(-\frac{1}{3}\varepsilon\right)\mathrm{i} + \left(\left(\frac{1}{3} + \sqrt{3}\right)\varepsilon\right)\mathrm{j} + \left(\left(\frac{4}{3} - \sqrt{3}\right)\varepsilon\right)\mathrm{k}.$$

The transformed point $\tilde{X}$ has coordinates $\left[\tilde{\Xi}_2, \tilde{\Xi}_3, \tilde{\Xi}_4\right]$, i.e. $\tilde{X} = \left[-\frac{1}{3}, \frac{1}{3} + \sqrt{3}, \frac{4}{3} - \sqrt{3}\right]$.

Remark 1. For a pure rotation, we take $\vec{v} = (0, 0, 0)$. We observe that we work with real quaternions in this case.

Remark 2. For a pure translation, we can take $\varphi = 0$. However, a pure translation is a trivial problem which is easier to calculate by simply adding a vector $\vec{v}$ to a point X. Therefore, it is not necessary to study the composition "translation, then rotation".

Remark 3. An appropriate translation also solves the situation where the axis of rotation passes through a general point, not necessarily the origin.

Remark 4. For further reading, we recommend [**1**].

References

1. M. J. Baker, `https://www.euclideanspace.com/`, section Dual Quaternions. Accessed: July 20, 2019.