Systems of homogeneous partial differential equations with constant coefficients — an instructive example

Miroslav Kureš

We will demonstrate how to solve systems of homogeneous partial differential equations with constant coefficients.

We solve

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial u}{\partial z} \tag{1}$$

$$\frac{\partial^3 u}{\partial y^3} = 2\frac{\partial^3 u}{\partial x^2 \partial z} \tag{2}$$

$$\frac{\partial^3 u}{\partial z^3} = 2\frac{\partial^3 u}{\partial x^2 \partial y} \tag{3}$$

for an unknown function $u(x, y, z)$. We associate the ideal

$$\mathfrak{i} = \langle x^3 - yz - z, y^3 - 2x^2 z, z^3 - 2x^2 y \rangle \tag{4}$$

in $\mathbb{C}[x, y, z]$ to this system. The polynomial belonging to the Gröbner basis of $\mathfrak{i}$ in the single variable z (in other words, belonging to the second eliminition ideal) is

$$z^{19} - 64z^{16} + 1280z^{13} - 4096z^{11} - 8192z^{10} + 8192z^7 + 4096z^3 \tag{5}$$

and one can decompose it by

$$z^3 \underbrace{\left(z^4 - 4z^3 + 4z^2 + 8\right)}_{(i)} \underbrace{\left(z^4 + 4z^3 + 12z^2 + 8\right)}_{(ii)} \underbrace{\left(z^4 - 8z^2 - 16z - 8\right)}_{(iii)} \underbrace{\left(z^4 + 8z^2 - 16z + 8\right)}_{(iv)}. \tag{6}$$

Now, we can find roots. First, the most left factor z^3 gives the triple root $z_{1,2,3} = 0$.

Further, the polynomial *(i)* has four imaginary roots $z_{4,5} = 1 + \sqrt{2} \pm \mathrm{i}$, $z_{6,7} = 1 - \sqrt{2} \pm \mathrm{i}$.

The polynomial *(ii)* has four imaginary roots $z_{8,9} = -1 + \sqrt{-1 + \sqrt{5}} \pm (1 - \sqrt{1 + \sqrt{5}})\,\mathrm{i}$, $z_{10,11} = -1 - \sqrt{-1 + \sqrt{5}} \pm (1 + \sqrt{1 + \sqrt{5}})\,\mathrm{i}$.

The polynomial *(iii)* has two real roots $z_{12,13} = \sqrt{2} \pm \sqrt{2 + 2\sqrt{2}}$ and two imaginary roots $z_{14,15} = -\sqrt{2} \pm \sqrt{-2 + 2\sqrt{2}}\,\mathrm{i}$.

The polynomial *(iv)* has four imaginary roots $z_{16,17} = -\sqrt{-1 + \sqrt{3}} \pm \left(\sqrt{2} + \sqrt{1 + \sqrt{3}}\right)\mathrm{i}$, $z_{18,19} = \sqrt{-1 + \sqrt{3}} \pm \left(\sqrt{2} - \sqrt{1 + \sqrt{3}}\right)\mathrm{i}$.

We can compute 17 distinct points of $\mathcal{V}(\mathfrak{i})$ in $\mathbb{C}^3$:

$$A_{1,2,3}: \quad x = 0, \quad y = 0, \quad z = 0$$

$$A_4: \quad \begin{aligned} x &= -1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,\mathrm{i}\\ y &= -1+\left(1+\sqrt{2}\right)\,\mathrm{i}\\ z &= 1+\sqrt{2}+\mathrm{i} \end{aligned}$$

$$A_5: \quad \begin{aligned} x &= -1-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\,\mathrm{i}\\ y &= -1-\left(1+\sqrt{2}\right)\,\mathrm{i}\\ z &= 1+\sqrt{2}-\mathrm{i} \end{aligned}$$

$$A_6: \quad \begin{aligned} x &= -1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\,\mathrm{i}\\ y &= -1+\left(1-\sqrt{2}\right)\,\mathrm{i}\\ z &= 1-\sqrt{2}+\mathrm{i} \end{aligned}$$

$$A_7: \quad \begin{aligned} x &= -1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,\mathrm{i}\\ y &= -1-\left(1-\sqrt{2}\right)\,\mathrm{i}\\ z &= 1-\sqrt{2}-\mathrm{i} \end{aligned}$$

$$A_8: \quad \begin{aligned} x &= -1+\frac{\sqrt{1+\sqrt{5}}}{4}+\frac{\sqrt{5+5\sqrt{5}}}{4}+\left(-3+\sqrt{5}\right)\frac{\sqrt{1+\sqrt{5}}}{4}\,\mathrm{i}\\ y &= 1-\sqrt{1+\sqrt{5}}+\left(1+\frac{\sqrt{1+\sqrt{5}}}{2}-\frac{\sqrt{5+5\sqrt{5}}}{2}\right)\,\mathrm{i}\\ z &= -1+\sqrt{-1+\sqrt{5}}+\left(1-\sqrt{1+\sqrt{5}}\right)\,\mathrm{i} \end{aligned}$$

$$A_9: \quad \begin{aligned} x &= -1+\frac{\sqrt{1+\sqrt{5}}}{4}+\frac{\sqrt{5+5\sqrt{5}}}{4}-\left(-3+\sqrt{5}\right)\frac{\sqrt{1+\sqrt{5}}}{4}\,\mathrm{i}\\ y &= 1-\sqrt{1+\sqrt{5}}-\left(1+\frac{\sqrt{1+\sqrt{5}}}{2}-\frac{\sqrt{5+5\sqrt{5}}}{2}\right)\,\mathrm{i}\\ z &= -1+\sqrt{-1+\sqrt{5}}-\left(1-\sqrt{1+\sqrt{5}}\right)\,\mathrm{i} \end{aligned}$$

$$A_{10}: \quad \begin{aligned} x &= -1-\frac{\sqrt{1+\sqrt{5}}}{4}-\frac{\sqrt{5+5\sqrt{5}}}{4}-\left(-3+\sqrt{5}\right)\frac{\sqrt{1+\sqrt{5}}}{4}\,\mathrm{i}\\ y &= 1+\sqrt{1+\sqrt{5}}+\left(1-\frac{\sqrt{1+\sqrt{5}}}{2}+\frac{\sqrt{5+5\sqrt{5}}}{2}\right)\,\mathrm{i}\\ z &= -1-\sqrt{-1+\sqrt{5}}+\left(1+\sqrt{1+\sqrt{5}}\right)\,\mathrm{i} \end{aligned}$$

$$A_{11}: \quad \begin{aligned} x &= -1-\frac{\sqrt{1+\sqrt{5}}}{4}-\frac{\sqrt{5+5\sqrt{5}}}{4}+\left(-3+\sqrt{5}\right)\frac{\sqrt{1+\sqrt{5}}}{4}\,\mathrm{i} \\ y &= 1+\sqrt{1+\sqrt{5}}-\left(1-\frac{\sqrt{1+\sqrt{5}}}{2}+\frac{\sqrt{5+5\sqrt{5}}}{2}\right)\mathrm{i} \\ z &= -1-\sqrt{-1+\sqrt{5}}-\left(1+\sqrt{1+\sqrt{5}}\right)\mathrm{i} \end{aligned}$$

$$A_{12}: \quad \begin{aligned} x &= 1+\sqrt{1+\sqrt{2}} \\ y &= \sqrt{2}+\sqrt{2+2\sqrt{2}} \\ z &= \sqrt{2}+\sqrt{2+2\sqrt{2}} \end{aligned}$$

$$A_{13}: \quad \begin{aligned} x &= 1-\sqrt{1+\sqrt{2}} \\ y &= \sqrt{2}-\sqrt{2+2\sqrt{2}} \\ z &= \sqrt{2}-\sqrt{2+2\sqrt{2}} \end{aligned}$$

$$A_{14}: \quad \begin{aligned} x &= 1-\sqrt{-1+\sqrt{2}}\,\mathrm{i} \\ y &= -\sqrt{2}+\sqrt{-2+2\sqrt{2}}\,\mathrm{i} \\ z &= -\sqrt{2}+\sqrt{-2+2\sqrt{2}}\,\mathrm{i} \end{aligned}$$

$$A_{15}: \quad \begin{aligned} x &= 1+\sqrt{-1+\sqrt{2}}\,\mathrm{i} \\ y &= -\sqrt{2}-\sqrt{-2+2\sqrt{2}}\,\mathrm{i} \\ z &= -\sqrt{2}-\sqrt{-2+2\sqrt{2}}\,\mathrm{i} \end{aligned}$$

$$A_{16}: \quad \begin{aligned} x &= 1+\frac{\sqrt{-1+\sqrt{3}}}{2}+\frac{\sqrt{-3+3\sqrt{3}}}{2}+\left(-1+\sqrt{3}\right)\frac{1+\sqrt{3}}{2}\,\mathrm{i} \\ y &= -\sqrt{-1+\sqrt{3}}-\left(\sqrt{2}+\sqrt{1+\sqrt{3}}\right)\mathrm{i} \\ z &= -\sqrt{-1+\sqrt{3}}+\left(\sqrt{2}+\sqrt{1+\sqrt{3}}\right)\mathrm{i} \end{aligned}$$

$$A_{17}: \quad \begin{aligned} x &= 1+\frac{\sqrt{-1+\sqrt{3}}}{2}+\frac{\sqrt{-3+3\sqrt{3}}}{2}-\left(-1+\sqrt{3}\right)\frac{1+\sqrt{3}}{2}\,\mathrm{i} \\ y &= \sqrt{-1+\sqrt{3}}+\left(\sqrt{2}+\sqrt{1+\sqrt{3}}\right)\mathrm{i} \\ z &= -\sqrt{-1+\sqrt{3}}-\left(\sqrt{2}+\sqrt{1+\sqrt{3}}\right)\mathrm{i} \end{aligned}$$

$$A_{18}: \qquad \begin{aligned} x &= 1-\frac{\sqrt{1-\sqrt{3}}}{2}-\frac{\sqrt{-3+3\sqrt{3}}}{2}-\left(-1+\sqrt{3}\right)\frac{1+\sqrt{3}}{2}\,\mathrm{i}\\ y &= -\sqrt{-1+\sqrt{3}}-\left(\sqrt{2}-\sqrt{1+\sqrt{3}}\right)\mathrm{i}\\ z &= \sqrt{-1+\sqrt{3}}+\left(\sqrt{2}-\sqrt{1+\sqrt{3}}\right)\mathrm{i}\end{aligned}$$

$$A_{19}: \qquad \begin{aligned} x &= 1-\frac{\sqrt{1-\sqrt{3}}}{2}-\frac{\sqrt{-3+3\sqrt{3}}}{2}+\left(-1+\sqrt{3}\right)\frac{1+\sqrt{3}}{2}\,\mathrm{i}\\ y &= -\sqrt{-1+\sqrt{3}}+\left(\sqrt{2}-\sqrt{1+\sqrt{3}}\right)\mathrm{i}\\ z &= \sqrt{-1+\sqrt{3}}-\left(\sqrt{2}-\sqrt{1+\sqrt{3}}\right)\mathrm{i}\end{aligned}$$

Now, we present basis functions of a certain class of solutions of the system.

For the real point $A_{1,2,3}$ we have the following real polynomial solutions:

$$\begin{gathered} 1, \quad x, \quad y, \quad x^2, \quad xy, \quad y^2, \quad x^3+6z, \quad xy^2, \quad x^4+24xz, \\ x^5+60x^2z+40y^3, \quad x^6+120x^3z+240xy^3+360z^2. \end{gathered} \tag{7}$$

For the real point A_{12} (and analogously for A_{13}) we have the following real exponential solution

$$\mathrm{e}^{\left(1+\sqrt{1+\sqrt{2}}\right)x+\left(\sqrt{2}+\sqrt{2+2\sqrt{2}}\right)y+\left(\sqrt{2}+\sqrt{2+2\sqrt{2}}\right)z}. \tag{8}$$

For imaginary points A_4 and A_5 (and analogously for all other pairs of complex conjugate roots) we have two complex exponential solutions, from which, by using Euler's identity, we can also obtain real solutions

$$\begin{gathered} \mathrm{e}^{\left(-1-\frac{1}{\sqrt{2}}\right)x-y+\left(1+\sqrt{2}\right)z}\cos\left(\frac{x}{\sqrt{2}}+\left(1+\sqrt{2}\right)y+z\right), \\ \mathrm{e}^{\left(-1-\frac{1}{\sqrt{2}}\right)x-y+\left(1+\sqrt{2}\right)z}\sin\left(\frac{x}{\sqrt{2}}+\left(1+\sqrt{2}\right)y+z\right). \end{gathered} \tag{9}$$